Scaling properties of scale-free evolving networks: Continuous approach

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The scaling behavior of scale-free evolving networks, arising in areas such as communications, scientific citations, collaborations, etc., is studied. We derive universal scaling relations describing properties of such networks, and indicate the limits of their validity. We show that the main properties of scale-free evolving networks may be described in the framework of a simple continuous approach. The simplest models of networks, growing according to a mechanism of preferential attachment of links to nodes, are used. We consider different forms of this preference, and demonstrate that the range of preferential attachments producing scale-free networks is wide. We also obtain scaling relations for networks with nonlinear, accelerating growth, and describe the temporal evolution of the arising distributions. Size effects—the cutoffs of these distributions—introduce restrictions for the observation of power-law dependences. Mainly we discuss the so-called degree distribution, i.e., the distribution of the number of connections of nodes. A scaling form of the distribution of links between pairs of individual nodes for a growing network of citations is also studied. We describe the effects of differences between nodes. The ''aging'' of nodes changes the exponents of the distributions. The appearance of a single node with high fitness changes the degree distribution of a network dramatically. If its fitness exceeds some threshold value, this node captures a finite part of all links of the network. We show that permanent random damage to a growing scale-free network—a permanent deletion of some links—radically changes the values of the scaling exponents. Results of other kinds of permanent damage are described.

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I. INTRODUCTION

One of the most impressive recent discoveries in the field of network evolution is the observation that a number of large growing networks are scale free, that is, the distributions of the number of connections of their nodes are of a power-law form [1–6]. Members of the class of scale-free networks are huge communications networks (the World Wide Web and the Internet [7,8]), networks of citations in the scientific literature [9,10], collaboration networks [11–13], some biological networks (nets of metabolic reactions [14]), etc. An incredible progression of science and information technology has produced networks, large enough (the World Wide Web contains about 10^9 nodes) to obtain reliable data.

Nevertheless, the experimental data are certainly not excellent. Indeed, the observation of such power laws is not easy. Arising distributions are very sensitive to size effects. Unfortunately, few really large growing networks are known as of yet, and many of the obtained data for not so huge nets are very preliminary. It is tempting to describe any decreasing experimental curve on a log-log plot by a linear fit. Nearly all the observations are for the most easily measurable quantity—degree distribution of nodes. (Following definitions of mathematicians and computer scientists, here we call the total number of connections of a node its *degree* [15]. If the links are directed, the number of incoming links of a node is called the *in-degree* of it, and the number of outgoing links the *out-degree*.) Information about others characteristics is much poorer.

Therefore, only a few scale-free growing networks have been observed. Why are these observations so important? Why is the interest in these networks so great? (In one of the last issues of Physical Review Letters, three papers, one after the other, were devoted to scale-free networks [16–18].) Of course, the reason for this is not only the power-law dependence of the distributions itself, but a variety of intriguing properties of scale-free networks which explain their existence in Nature. Degree distribution is a simple, but very important, characteristic of a network. In particular, it was shown that networks with power-law degree distributions are extremely resilient against random breakdowns if the γ exponent of their degree distribution does not exceed 3 [16]. This property is vitally important for communications and biological networks. One also has to point out the following circumstance. A power-law dependence of the degree distribution indicates a scale invariance of the characteristics of networks and their hierarchically organized structure. An investigation of the scaling properties of scale-free networks, and the connections between them, is a topic of the present paper.

We study the formulated problem by applying two different approaches in arbitrary order. First, we obtain universal scaling relations using general considerations. Second, we demonstrate features of the scaling behavior of networks using minimal models providing the effect being studied. At the moment, the only known mechanism producing scale-free networks is preferential attachment—new links are preferentially attached to nodes with a large number of connec-

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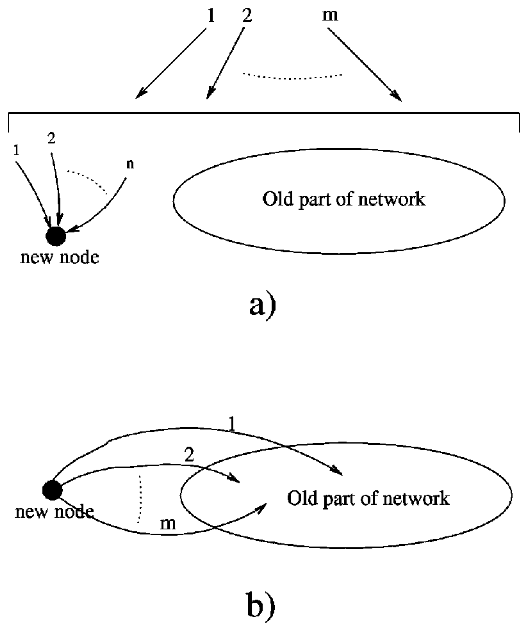

FIG. 1. Scheme for the growth of the two standard network models considered in the present paper. (a) At each time step, a new node with *n* incoming links is added. Source ends of all links are placed anywhere. In addition, *m* links are distributed preferentially among all nodes. This means that a target end of each of these links is attached to some node according to a rule of preference. (b) A citation network model. $n=0$. At each increment of time, a new node with *m* outgoing links is added. Target ends of these links are distributed preferentially among old nodes.

tions (large degree) [5,19,20]. The probability that a new link turns out to be attached to a node is a function of its degree. Related ideas were discussed long ago by Simon [21]. All these models belong to a class of stochastic multiplicative processes [22].

All the models used can be described by linear master equations [18] which are solvable exactly in several particular cases. In this paper, we treat them using a trivial continuous approach [19,20,18] that, as we show, describes the main features of the networks and produces exact values of the exponents. A demonstration of the possibilities of the continuous approach is one of the aims of our paper.

Here, we use several different models of evolving networks (with undirected and directed links), but the skeleton of all them is the same (see Fig. 1). At each increment of time, a new node is added. Therefore, the number of the last node, *t*, may be called ''time.'' Each node is labeled by the time of its birth, $s=0,1,2,\ldots\leq t$. Together with a new node, several new links is appeared in the network. A part of these links is distributed between nodes preferentially according to their degrees (we notate them k) or in-degrees ($\equiv q$). (Out-degree distributions are discussed only in one place in the paper.) In general considerations, k and q will be equivalent if no special notion will be made.

The following ''one-node'' characteristics may be introduced. $p(k,s,t)$ is the probability that the node s has a degree k at time t. $P(k,t)$ is the total degree distribution of a network in time t. $P(k)=P(k,t\to\infty)$ is the stationary distribution, γ is the corresponding scaling exponent, and $P(k)\propto k^{-\gamma}$ for large k. $\overline{k}(s,t)$ is the average degree of the node s at time t. β is its scaling exponent; $\overline{k}(s,t)\propto s^{-\beta}$. The corresponding definitions for in-degree distributions are similar.

In Sec. II we introduce the continuous approach. Considering a simple example we explain the reasons for using this approach, and derive useful relations. In Sec. III, for linearly growing scale-free networks (the total number of links is proportional to a total number of nodes with a time-independent coefficient), using general considerations, we obtain scaling forms of involved quantities and universal relations between scaling exponents. Section IV is devoted to a study of different types of preferential attachment. Using the continuous approach, we describe (i) the simplest network in which new links are attached without any preference (the produces exponential distributions), (ii) a linear type of preferential attachment (that produces scale-free networks with $2<\gamma<\infty$), (iii) a mixture of preferential and random linking (that also produces scale-free networks with $2<\gamma<\infty$), and (iv) more general cases also producing scale-free networks. In fact, in Secs. IV, V, and VI, different realizations of a linear preference function $G(s,t)k+A(s,t)$, according to which new links are distributed among nodes, are studied. In Sec. V, the ''fitness'' of a node, $G(s,t)$, depends only on its age, $G(s,t)\propto(t-s)^{-\delta}$. This changes the scaling exponents. In Sec. VI, the fitness of nodes depends only on dates of their birth, *s*. In this case, the network may exhibit an intriguing phenomenon of ''condensation'' of a finite fraction of links on the fittest node that was quite recently reported [23] (also see Ref. [24]). Nevertheless, we demonstrate that the network remains scale free.

Note that, in most models considered here, new links may also connect old nodes. Nevertheless, in Sec. VII, we study a specific growing network in which new connections are possible only between the new and old nodes, that, in particular, may be used to describe networks of scientific citations. In this simple situation, using the continuous approach, it is possible to describe distributions of links between pairs of individual nodes.

Nonlinearly growing networks are considered in Sec. VIII. For this more general case, we obtain scaling relations, and describe nonstationary distributions $P(q,t)$ and their cutoffs impeding observation of power-law dependences for finite-size networks.

In Sec. IX, we study influence of *permanent damage* on the scaling characteristics of growing networks. We show that such a type of damage produces a much stronger effect on the network than the previously studied instant damage [16,25–28]. The high quality of the continuous approach is discussed in Sec. X using already known exact results.

II. REASONS FOR THE CONTINUOUS APPROACH

We start from one of the simplest models of growing networks with preferential attachment, proposed by Barabási and Albert [5], which belongs to a class of more general models which were solved exactly afterwards [17,18]. At each increment of time, a node is added. It connects to one of the old nodes chosen with a probability proportional to the degree of this old node, i.e., to the total number of its connections. Using the notations introduced in Sec. I, it is possible to immediately write a master equation for the degree distribution of individual nodes (see Ref. [18]),

$$p(k,s,t+1)=\frac{k-1}{2t}p(k-1,s,t)+\left(1-\frac{k}{2t}\right)p(k,s,t), \tag{1}$$

with, e.g., $t=1,2,3,\ldots$ and $s=0,1,2,\ldots,t$. Hence, at time $t=1$, a pair of connected nodes $s=0$ and 1 is present. Therefore, the initial condition is $p(k,s=0,1,t=1)=\delta_{k,1}$, and $p(k,t,t)=\delta_{k,1}$ is the boundary condition. Equation (1) may be rewritten in the form

$$2t[p(k,s,t+1)-p(k,s,t)]=(k-1)p(k-1,s,t) -kp(k,s,t). \tag{2}$$

Passing to the continuous limit in t and k, we obtain

$$2t\frac{\partial p(k,s,t)}{\partial t}+\frac{\partial[kp(k,s,t)]}{\partial k}=0 \tag{3}$$

and

$$\frac{\partial[kp(k,s,t)]}{\partial \ln\sqrt{t}}+\frac{\partial[kp(k,s,t)]}{\partial \ln k}=0. \tag{4}$$

The solution of Eq. (4) is $kp(k,s,t)=\delta(\ln k-\ln\sqrt{t/s}+\text{const})$. The boundary condition is fulfilled if the solution is of the following form:

$$p(k,s,t)=\delta(k-\sqrt{t/s}). \tag{5}$$

Therefore, we see that the transition to the continuous limit in the master equation leads to a δ-function form of degree distributions of individual nodes.

The main quantity of interest is the total degree distribution of the entire network:

$$P(k,t)=\frac{1}{t+1}\sum_{s=0}^{t}p(k,s,t). \tag{6}$$

In the continuous approximation, the stationary degree distribution is of the form

$$P(k)=P(k,t\to\infty)=\lim_{t\to\infty}\frac{1}{t}\int_0^t ds\,p(k,s,t). \tag{7}$$

Inserting the obtained expression for $p(k,s,t)$ [Eq. (5)] into Eq. (7), one obtains the continuous approximation result for this model: $P(k)=2/k^3$ [5,19].

Another way to obtain this expression is to derive an equation for the total degree distribution, $P(k,t)$. Applying $\Sigma_{s=0}^{t}$ to both sides of Eq. (1), and passing to the $t\to\infty$ limit, one obtains

$$P(k)+\tfrac{1}{2}[kP(k)-(k-1)P(k-1)]=\delta_{k,1}. \tag{8}$$

In the continuous limit, this equation is of the form $P(k)+(1/2)d[kP(k)]/dk=0$. Its solution is $P(k)\propto k^{-3}$.

Often, it is convenient to proceed in a slightly different way. Let us introduce the average degree of an individual node:

$$\overline{k}(s,t)=\sum_{k=1}^{\infty}kp(k,s,t)=\int_0^{\infty}dk\,k\,p(k,s,t). \tag{9}$$

Applying $\int_0^{\infty}dk\,k$ to Eq. (3), and integrating its right hand side by parts, we obtain

$$\frac{\partial\overline{k}(s,t)}{\partial t}=\frac{\overline{k}(s,t)}{\int_0^t du\,\overline{k}(u,t)}. \tag{10}$$

Here we used an obvious equality—the total number of links in this network, $\int_0^t ds\int_0^{\infty}dk\,k\,p(k,s,t)$, is equal to $2t$. A boundary condition for Eq. (10) is $\overline{k}(t,t)=1$. It is possible to use equations similar to Eq. (10), and not to study each time corresponding master equation.

The meaning of Eq. (10) is quite obvious—each new link is distributed homogeneously among all nodes, taking account of the particular form of preference. Initially, this approach has been called ''mean field'' [5,19], but later it was shown that it is a continuous approximation [20,18].

One should note that we pass to the continuous limit for both variables k and t. The continuous limit for t is not dangerous if we consider sufficiently large networks and do not study some peculiarities of distributions related to the particular form of the initial conditions (for a very simple model of a scale-free network, an exact solution was found for all values of k and t [29]). The continuous approximation in k needs more care, so we discuss its quality throughout the paper.

III. SCALING RELATIONS FOR SCALE-FREE NETWORKS

In the continuous approach, a knowledge of the average degree of nodes, $\overline{k}(s,t)$, lets us obtain the total degree distribution $P(k,t)$. Here we do not restrict ourselves to any particular model, and our general results are also valid for in-degree q (and out-degree) distributions. Nevertheless, in any case, $\overline{q}(s,t)$ is the solution of an equation similar to Eq. (10).

Indeed, one can write

$$P(k,t)=\frac{1}{t}\int_0^t ds\,\delta(k-\overline{k}(s,t)) =-\frac{1}{t}\left(\frac{\partial\overline{k}(s,t)}{\partial s}\right)^{-1}[s=s(k,t)], \tag{11}$$

where $s(k,t)$ is a solution of the equation $k=\overline{k}(s,t)$. Now we can easily connect involved quantities. Assuming that $P(k)$ and $\overline{k}(s,t)$ exibit scaling behavior, that is, $P(k)\propto k^{-\gamma}$ for large k and $\overline{k}(s,t)\propto s^{-\beta}$ for $1\ll s\ll t$, one obtains $s\propto k^{-1/\beta}$ and $k^{-\gamma}\propto\partial s/\partial k\propto k^{-1-1/\beta}$. Therefore, $\gamma=1+1/\beta$, and we obtain the scaling relation

$$\beta(\gamma-1)=1. \tag{12}$$

Let us show that Eq. (12) is universal. Here we can proceed using general considerations. In the present section, we study only linearly growing networks (the input flow of new links does not depend on time) which produce stationary degree distributions at long times. More complex cases will be considered in Sec. VIII A.

Hence let $P(k)$ be stationary; it then follows from Eq. (7) that $p(k,s,t)$ has to be of the form $p(k,s,t)=\rho(k,s/t)$. The normalization condition is

$$\int_0^\infty dk p(k,s,t)=1, \tag{13}$$

so $\int_0^\infty dk\rho(k,x)=1$. From this equation, it follows that $\rho(k,x)=g(x)f(kg(x))$, where $g(x)$ and $f(x)$ are arbitrary functions. From the definition of $\overline{k}(s,t)$ [Eq. (9)], using its scaling behavior, we obtain $\int_0^\infty dk k\rho(k,x)\propto x^{-\beta}$. Substituting $\rho(k,x)$ into this relation, one obtains that $g(x)\propto x^{\beta}$. Of course, without loss of generality, one may set $g(x)=x^{\beta}$, so we obtain the following scaling form of the degree distribution of individual nodes:

$$p(k,s,t)=(s/t)^{\beta}f(k(s/t)^{\beta}). \tag{14}$$

Finally, assuming a scaling behavior of $P(k)$, i.e., $\int_0^\infty dx\rho(k,x)\propto k^{-\gamma}$, and using Eq. (14), we obtain $\gamma=1+1/\beta$, i.e., Eq. (12) is universal. Here we used the fast convergence of $\rho(k,x)$ at large x (it follows from our exact results [18]). One should note that, during this derivation, we did not use any approximations.

IV. TYPES OF PREFERENTIAL ATTACHMENT

A. Absence of preference

We need an example of a non-scale-free network for comparison, so we start with the simplest growing network with random attachment of new links introduced in Refs. [5,19]. Again, as in Sec. II, at each increment of time, a new node is added to the network. Now it connects with a randomly chosen (i.e., without any preference) old node. The growth begins from a configuration consisting of two connected nodes at time $t=1$. This model is even simpler than the well-known Erdös-Rényi model [30] (also see Ref. [31]), and its exact solution is trivial. The master equation describing the evolution of the degree distribution of individual nodes can be written as

$$p(k,s,t+1)=\frac{1}{t+1}p(k-1,s,t)+\left(1-\frac{1}{t+1}\right)p(k,s,t), \tag{15}$$

where $p(k,s=0,1,t=1)=\delta_{k,1}$ and $\delta(k,t,t\geq 1)=\delta_{k,1}$. Applying $\Sigma_{s=0}^{t}$ to both sides of Eq. (15), and using the definition of total degree distribution [Eq. (6)], we obtain

$$(t+1)P(k,t+1)-tP(k,t)=P(k-1,t)-P(k,t)+\delta_{k,1}. \tag{16}$$

The corresponding stationary equation,

$$2P(k)-P(k-1)=\delta_{k,1}, \tag{17}$$

has a solution of an exponential form: $P(k)=2^{-k}$. Therefore, networks of such type are often called ‘‘exponential.’’

From Eq. (15), one may also find the degree distribution of individual nodes, $p(k,s,t)$, for large s and t and fixed s/t:

$$p(k,s,t)=\frac{s}{t}\frac{1}{(k+1)!}\ln^{k+1}\left(\frac{t}{s}\right). \tag{18}$$

One sees that this function decreases rapidly at large values of degree, k.

Thus degree distributions of networks growing without preferential attachment differ strikingly from distributions of scale-free networks described in Sec. III. Note that the singularity of $p(k,s,t)$ [Eq. (18)] at $s/t\to 0$ is much weaker than for scale-free networks.

B. Linear preference

Let us demonstrate the continuous approach in detail, using a more general model than in Sec. II, which produces a wide range of γ exponent values [18] [see Fig. 1(a)]. Let us consider a network with directed links. Here we study the distribution of the numbers of *incoming* links of nodes (*in-degree*, q).

At each time step a new node is added. It has n incoming links. These links go out from arbitrary nodes or even from some external source. Simultaneously, m extra links are distributed with preference. This means, again that they go out from nonspecified nodes or from an external source, but that a target end of each of them is attached to a node chosen preferentially: the probability to choose some particular node is proportional to a function of its in-degree, we call this a *preference function*. In the simple model that we consider in the present section, this probability is proportional to $q+A$, where A is a constant which we call *additional attractiveness*. We shall see that its reasonable values are $A>-n$.

In the continuous approach, we may assume that m and n are not necessarily integer numbers, but any positive numbers. We do not worry about the source ends of links because here we study only in-degree distributions.

Then the equation for the average in-degree is of the form

$$\frac{\partial\overline{q}(s,t)}{\partial t}=m\frac{\overline{q}(s,t)+A}{\int_0^t du[\overline{q}(u,t)+A]}, \tag{19}$$

with the initial condition, $\overline{q}(0,0)=0$, and the boundary one, $\overline{q}(t,t)=n$. Applying $\int_0^t ds$ to Eq. (19), we obtain

$$\frac{\partial}{\partial t}\int_0^t ds\overline{q}(s,t)=\int_0^t ds\frac{\partial}{\partial t}\overline{q}(s,t)+\overline{q}(t,t), \tag{20}$$

so

$$\int_0^t ds\overline{q}(s,t)=(m+n)t. \tag{21}$$

This relation is quite natural—the total in-degree of the network is equal to the total number of created links, and we see that Eq. (19) is consistent. Therefore, Eq. (19) takes the form

$$\frac{\partial\overline{q}(s,t)}{\partial t}=\frac{m}{m+n+A}\frac{\overline{q}(s,t)+A}{t}. \tag{22}$$

Its general solution is

$$\overline{q}(s,t)+A=C(s)t^{m/(m+n+A)}, \tag{23}$$

where $C(s)$ is an arbitrary function of s. Accounting for the boundary condition $\overline{q}(t,t)=n$, one has

$$\overline{q}(s,t)+A=(n+A)\left(\frac{s}{t}\right)^{-m/(m+n+A)}. \tag{24}$$

Hence the scaling exponents are

$$\beta=\frac{m}{m+n+A} \tag{25}$$

and

$$\gamma=2+\frac{n+A}{m}. \tag{26}$$

One sees that, for $n+A>0$, the exponent γ is in the range $2<\gamma<\infty$, while β belongs to the interval $(0,1)$. We see that n plays the same role as A.

If we set $n=0$, and demand that all new links have to go out from new nodes, we obtain a network of citations [see Fig. 1(b)]. Note that, when $A=m$ and $n=0$, we obtain the particular case of the Barabási and Albert's model [5] in which each new link is connected with a new node. Indeed, in this case, degree and in-degree of nodes are coupled rigidly, $k=q+m$, and the obtained results are valid for the degree distributions. If, in addition, we set $m=1$, we obtain the model considered in Sec. II.

C. Mixture of preferential and random attachment

Now we can consider a slightly more complex model. We will demonstrate that scale-free networks may be obtained even without a ''pure'' preferential attachment. We discuss the model introduced in Sec. III but with one new element. In addition, at each time step we allow, n_r links to be distributed between nodes randomly, without any preference. Again, we study in-degree distributions, so these links may go out from anywhere, but their target ends are attached to randomly chosen nodes. Then

$$\frac{\partial\overline{q}(s,t)}{\partial t}=\frac{n_r}{t}+m\frac{\overline{q}(s,t)+A}{\int_0^t du[\overline{q}(u,t)+A]}. \tag{27}$$

Again, $\overline{q}(0,0)=0$ and $\overline{q}(t,t)=n$. In this case, $\int_0^t ds\overline{q}(s,t)=(n_r+m+n)t$, so Eq. (27) takes the form

$$\frac{\partial\overline{q}(s,t)}{\partial t}=\frac{m}{n_r+m+n+A}\times\frac{1}{t}\left[\overline{q}(s,t)+A+\frac{n_r}{m}(n_r+m+n+A)\right]. \tag{28}$$

Its solution, with account for the boundary condition, is

$$\overline{q}(s,t)+A+\frac{n_r}{m}(n_r+m+n+A)=\frac{n_r+m}{m}(n+n_r+A)\left(\frac{s}{t}\right)^{-m/(n_r+m+n+A)}. \tag{29}$$

Here m, n, and n_r are positive numbers, and $n_r+n+A>0$.

Therefore, we obtain a scale-free network, with the exponents

$$\beta=\frac{m}{m+n_r+n+A},\qquad \gamma=2+\frac{n_r+n+A}{m}. \tag{30}$$

Thus an additional fraction of randomly distributed links does not delete a power-law dependence of the distributions, but only increases γ.

In two computer-science papers [3,32], a similar model of a network with directed links was considered. At each time step, a new node is added to the net. It has one outgoing link. The other end of this link is attached to one of the old nodes by the following rule. (i) With probability p, it is attached to a random node. (ii) With probability $1-p$, it finds a random node and attaches itself to its sole target neighbor node. (In this particular case, this is the same as choosing a random link and connecting a new node with its target end.)

One can see that this model corresponds to the particular case, $n=0$, $n_r=p$, $m=1-p$, and $A=0$ of the network that we consider in the present section. From Eq. (30), we obtain $\gamma=2+p/(1-p)=1+1/(1-p)$. Note that, in Refs. [3,32], the wrong result was presented. The necessary additional unit is absent there. Indeed, if one of the factors (random attachment) produces $\gamma=\infty$ and another one (preferential attachment) produces $2<\gamma<\infty$, then their interplay cannot produce $\gamma<2$.

It is worth emphasizing here the close connection of the models that we consider with the well-known Simon's model [21], discussed in relation with networks in Ref. [33]. In Simon's model, if one uses the terminology of growing networks with directed links, at each time step a new link appears. Since here we discuss only in-degree, it is again not important from where it goes out. With some fixed probability, a new node is created, and the target end of the link is attached to it. With a complementary probability, the target end of the link is attached to the target end of a randomly chosen old node. Then only one point in Simon's model differs from the models that we consider in the present paper: at each increment of time, a link is added, but not the node as in our case. Of course, this does not influence the results for large networks and the scaling exponents.

D. WWW exponents

The previous result for the γ exponent, [Eq. (30)] allows us to obtain the crudest estimates for the exponents of the World Wide Web (WWW) [33,34] which are measured with sufficient precision for comparison. Let us apply the model of Sec. IV C to the growth of the World Wide Web. This means that we have to assume that each time a new node of the Web appears, on average, the same number of new links arises between its nodes. We neglect many very important factors, including the elimination of some nodes and links during the growth, etc.

We do not know the values of the quantities on the left hand side of Eq. (30). The constant A may take *any* values between $-(n_r+n)$ and infinity; the number of randomly distributed links, n_r, in principle, may be not small (there exist many individuals making their references practically at random), and n is not fixed. From the experimental data [4], we know more or less the sum, $m+n+n_r\sim 10\gg 1$ (between 7 and 10, more precisely), and that is all.

The only thing we can do is to fix the scales of the quantities. The natural characteristic values for n_r+n+A in Eq. (30) are (a) 0, (b) 1, (c) $m\gg 1$, and (d) infinity. In the first case, each node has a zero initial attractiveness, and all new links are directed to the oldest node, $\gamma\to 2$. In the last case, there is no preferential attachment, and the network is not scale free, $\gamma\to\infty$. Let us consider the truly important cases (b) and (c).

(b) How do pages appear on the Web? Suppose you want to create your own personal home page. Of course, first you prepare it, put in references, etc. But that is only the first step. You have to make it accessible on the Web, to launch it. You come to your system administrator. He places a reference to it (usually one reference) in the home page of your institution, and that is more or less all—your page is on the World Wide Web. There is another way of appearing of new documents on the Web. Imagine you already have your personal home page and want to launch a new document. The process is even simpler than the one described above. You simply insert at least one reference to the document into your page, and that is enough for the document to be included on the World Wide Web. If the process of the appearance of each document on the Web is as simple as the creation of your page—only one reference to the new document ($n=1$)—and if one forgets about the terms n_r and A in Eq. (30), then, for the γ_{in} exponent of the distribution of the incoming links (in-degree distribution), we immediately obtain the estimation $\gamma_{in}-2\sim 1/m\sim 10^{-1}$. This estimation coincides with the experimental value $\gamma_{in}-2=0.1$ [4]. (Here we have introduced the notation γ_{in}, because in the present section we discuss different distributions.) Therefore, the estimation seems to be good. Nevertheless, we should again repeat that this estimation follows only from the fixation of scales of the involved quantities. We emphasize that there are no any general reasons to set, e.g., $A=0$. Many real processes are not included in this estimation. The aging of nodes changes γ (see Sec. V and Ref. [20]). The permanent disappearance of links (see Ref. [35] and Sec. IX B) (the half-life of a page on the Web is of the order of half a year) also changes γ. The ratio between the total number of links and the number of nodes in the Web is not constant [4], it increases with time, and the growth of the Web is nonlinear. This factor also changes the value of γ; in the future it may become even lower than 2; see Ref. [36] and Sec. VIII.

(c) Above we discussed the distribution of incoming links. Equation (30) may be also applied for the distribution of links which come out from documents of the Web, since the model of Sec. III may be reformulated for outgoing links of nodes. In this case all the quantities in Eq. (30) take other values which are again unknown. Nevertheless, one may think that the number of links distributed without any preference, n_r, is not small. Indeed, even beginners proceed by linking their pages. Also, usually there are several references, n, in each new document. Hence $n+n_r\sim m$—we have no another available scale—and $\gamma_{out}-2\sim m/m\sim 1$. We again can compare this estimation with the experimental value: $\gamma_{out}-2=0.7$ [4].

E. Generalized form of preference

What other forms of preference produce scale-free networks? Let us list the main possibilities. In the present section and in Secs. V and VI, we consider only a linear form of a preference function. Nonlinear preference functions are discussed in Sec. X.

A reasonable (linear) form of the probability for a new link attached to a node s at time t is $p_{s,t}=G_{s,t}k_{s,t}+A_{s,t}$. The coefficient $G_{s,t}$ may be called the *fitness* of a node [23,37], and $A_{s,t}$ is *additional attractiveness*. One can consider the following particular cases.

(i) $G=\text{const}, \quad A=A(s)$. In this case, the additional attractiveness $A(s)$ may be treated as ascribed to individual nodes. A possible generalization is to make it a random quantity. One can check that the answers do not change crucially—one only has to substitute the average value $\overline{A}$, instead of A, into the previous expressions for scaling exponents. Note that n and m may also be random, and substituted with $\overline{n}$ and $\overline{m}$ in the expressions for the exponents.

There is a more interesting possibility: constructing a direct generalization of the network considered in Sec. IV C. For this, we may ascribe the additional attractiveness not to nodes but to new links and again make this a random quantity. Therefore, new links play the role of fans with different degrees of passion for their idols (nodes). This is case (ii), $G=\text{const}, \quad A=A(t)$, where $A(t)$ is random.

Let a distribution function of A be $P(A)$. Then our main equation looks like

$$\frac{\partial \overline{q}(s,t)}{\partial t}=\overline{m}\int dAP(A)\frac{\overline{q}(s,t)+A}{\int_0^t du[\overline{q}(u,t)+A]}. \quad (31)$$

The initial and boundary conditions are $\overline{q}(0,0)=0$ and $\overline{q}(t,t)=\overline{n}$. Hence $\int_0^t ds\overline{q}(s,t)=(\overline{n}+\overline{m})t$. Therefore,

$$\frac{\partial \overline{q}(s,t)}{\partial t}=\frac{\overline{m}}{t}\left[\overline{q}(s,t)\int\frac{dAP(A)}{\overline{n}+\overline{m}+A}+\int\frac{dAAP(A)}{\overline{n}+\overline{m}+A}\right]. \quad (32)$$

Thus we again obtain a scale-free network with the exponents

$$\beta=\overline{m}\int\frac{dAP(A)}{\overline{n}+\overline{m}+A} \quad (33)$$

and

$$\gamma=1+\left[\int\frac{dAP(A)}{1+(\overline{n}+A)/\overline{m}}\right]^{-1}. \quad (34)$$

These expressions generalize the corresponding results of Secs. IV B and IV C. One can see that the values of the γ exponent are again between 2 and ∞, β is between 0 and 1.

(iii) $A=\text{const}$. The case $G(s,t)\propto(s-t)^{-\alpha}$, the aging of nodes, is considered in Sec. V, it produces $2<\gamma<\infty$. Different versions of the s-dependent fitness, $G(s,t)=G(s)$, are studied in Sec. VI. The case of the power-law dependence $G(s)\propto s^{-\Delta}$ is considered in Sec. VI A. Three different types of behavior are possible: $\Delta<0$, an exponential network; $\Delta=0$, a scale-free network; and $\Delta>0$, in which the oldest node receives a finite part of the total degree of a network.

One can consider a network in which several nodes are "stronger" than others. We investigate effects arising in this situation in Sec. VI B. A homogeneous mixture of nodes with two different values of fitness is studied in Sec. VI C.

The case of fluctuating $G(s)$ was considered by Bianconi and Barabási [37]. When the distribution of G is homogeneous, i.e., G is homogeneously distributed between two fixed values, the distribution has a logarithmic factor $P(q)\propto q^{-\gamma}/\ln q$, where $\gamma>2$. For some special forms of the distribution of G, strong cooperative effects were found recently [23].

(iv) $A=\text{const}$, $G=G(t)$. This case is reduced to case (ii).

V. EFFECT OF AGING OF NODES

How does the structure of the network change if one introduces an aging of the sites [20], i.e., if the probability of the connection of the new node with some old one is proportional not only to the degree of the old node but also to the power of its age ($\tau^{-\alpha}$, for example)? Here we introduce the aging exponent α. Such aging is natural for networks of scientific citations in which the old papers usually have low attractiveness.

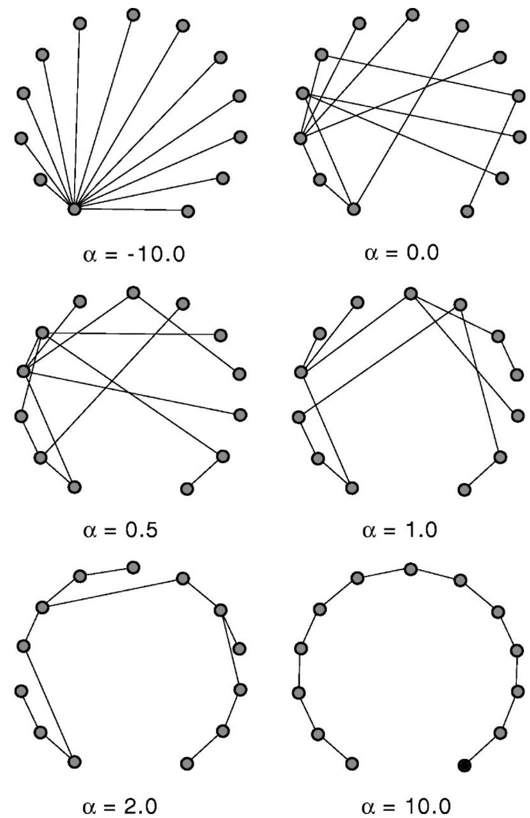

FIG. 2. Change of the structure of the network with an aging of nodes with an increase of the aging exponent α. The aging is proportional to $\tau^{-\alpha}$, where τ is the age of the site. The network grows clockwise starting from the site below on the left. Each time, one new site with one link is added.

The simplest analytical expressions may be obtained for the model of Sec. II: links are undirected, and every new node is connected by a single link with some old node which is chosen according to the same rule of preference as in Sec. II. Figure 2 demonstrates that the structure of the network depends crucially on the value of the aging exponent. When α is large enough (larger than 1), the network looks like a linear structure. For low negative values of α, i.e., for large $-\alpha$ (in this case, the attractiveness of the documents increases with the growth of their age) many links are attached to the oldest nodes.

The resulting equation for such a network is of the form

$$\frac{\partial \overline{k}(s,t)}{\partial t}=\frac{\overline{k}(s,t)(t-s)^{-\alpha}}{\int_0^t du\overline{k}(u,t)(t-u)^{-\alpha}}, \quad \overline{k}(t,t)=1. \quad (35)$$

Then $\int_0^t ds\overline{k}(s,t)=2t$.

We search for the solution of Eq. (35) in a scaling form:

$$\overline{k}(s,t)\equiv\kappa(s/t), \quad s/t\equiv\xi. \quad (36)$$

Then Eq. (35) becomes

$$-\xi(1-\xi)^{\alpha}\frac{d\ln\kappa(\xi)}{d\xi}=\left[\int_0^1 d\zeta\kappa(\zeta)(1-\zeta)^{-\alpha}\right]^{-1}\equiv\beta,$$
$$\kappa(1)=1, \tag{37}$$

where β is a constant which is as yet unknown. One can understand that this is just the exponent β of the average degree of individual nodes, since, in the scaling region $\xi \ll 1$, Eq. (37) provides the power-law dependence $\kappa(\xi) \propto \xi^{-\beta}$. We also obtain the relation $\int_0^1 d\zeta\kappa(\zeta)=2$. Our aim is to find β. For this, we have to find a solution of Eq. (37) containing this unknown parameter β, and substitute it into its definition, the left part of Eq. (37), or into the relation for the total number of links.

The solution of Eq. (37) is

$$\kappa(\xi)=B\exp\left[-\beta\int\frac{d\xi}{\xi(1-\xi)^{\alpha}}\right], \tag{38}$$

where B is a constant. The indefinite integral in Eq. (38) may be taken as

$$\int\frac{d\xi}{\xi(1-\xi)^{\alpha}}=\ln\xi+\sum_{j=0}^{\infty}\frac{1}{j!(j+1)^2}\alpha \times(\alpha+1)\dots(\alpha+j)\xi^{j+1} =\ln\xi+\alpha\,{}_3F_2(1,1,1+\alpha;2,2;\xi), \tag{39}$$

where ${}_3F_2()$ is the hypergeometric function. Recalling the boundary condition $\kappa(1)=1$, we find the constant B. Thus the solution is

$$\kappa(\xi)=e^{-\beta[C+\psi(1-\alpha)]}\xi^{-\beta} \times\exp[-\beta\alpha\xi\,{}_3F_2(1,1,1+\alpha;2,2;\xi)], \tag{40}$$

where $C=0.5772\dots$ is Euler's constant, and $\psi()$ is the ψ function. The transcendental equation for β may be written if one substitutes Eq. (40) into the right hand side of Eq. (37):

$$\beta^{-1}=e^{-\beta[C+\psi(1-\alpha)]}\int_0^1\frac{d\zeta}{\zeta^{\beta}(1-\zeta)^{\alpha}} \times\exp[-\beta\alpha\zeta\,{}_3F_2(1,1,1+\alpha;2,2;\zeta)]. \tag{41}$$

[An equivalent equation one may obtain substituting the solution, Eq. (40), into the relation $\int_0^1 d\zeta\kappa(\zeta)=2$]. The γ exponent may be obtained using the universal relation, Eq. (12).

The solution of Eq. (41) exists in the range $-\infty<\alpha<1$. One may also find simple expressions for $\beta(\alpha)$ and $\gamma(\alpha)$ at $\alpha\to 0$:

$$\beta\cong\frac{1}{2}-(1-\ln 2)\alpha, \quad \gamma\cong 3+4(1-\ln 2)\alpha, \tag{42}$$

where the numerical values of the coefficients are $1-\ln 2 =0.3069\dots$ and $4(1-\ln 2)=1.2274\dots$. We used the relation ${}_3F_2(1,1,1;2,2;\zeta)=Li_2(\zeta)/\zeta\equiv(\sum_{k=1}^{\infty}\zeta^k/k^2)/\zeta$ while deriving Eq. (42). Here $Li_2()$ is the polylogarithm function of order 2.

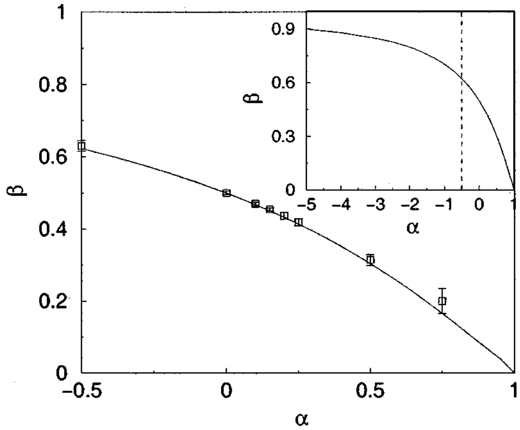

FIG. 3. β exponent of the mean degree vs the aging exponent α. Points are obtained from simulations [20]. The line is the solution of Eq. (41). The inset shows the analytical solution in the range $-5<\alpha<1$. Note that $\beta\to 1$ if $\alpha\to-\infty$.

In the limit of $\alpha\to 1$, using the relation ${}_3F_2(1,1,2;2,2;\zeta)=-\ln(1-\zeta)/\zeta$, we find

$$\beta\cong c_1(1-\alpha), \quad \gamma\cong\frac{1}{c_1}\frac{1}{1-\alpha}. \tag{43}$$

Here $c_1=0.8065\dots$ and $c_1^{-1}=1.2400\dots$: the constant c_1 is the solution of the equation $1+1/c_1=\exp(c_1)$. The dependences $\beta(\alpha)$ and $\gamma(\alpha)$ are shown in Figs. 3 and 4. For comparison, data of simulations [20] are also plotted. One sees that $\beta\to 1$ and $\gamma\to 2$ in the limit $\alpha\to-\infty$. Therefore, the whole range of the variation of β is (0,1), and that of γ is $(2,\infty)$. One should note that, in this section, we studied the case of $m=1$ but the results, the scaling exponents, are, of course, independent of m.

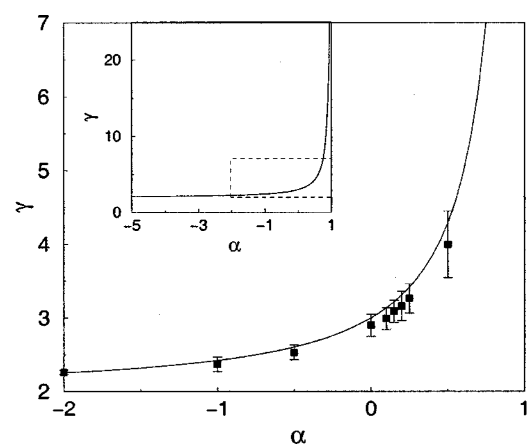

FIG. 4. γ exponent of the degree distribution vs the aging exponent α. The points show the results of simulations [20]. The line is the solution of Eq. (41), taking account of Eq. (12). The inset depicts an analytical solution in the range $-5<\alpha<1$.

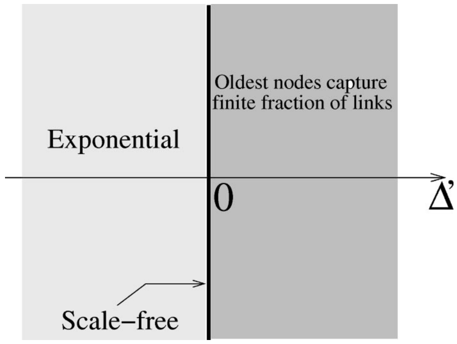

FIG. 5. Phases of the network with varying strengths of the nodes, $G(s)=s^{-\Delta'}$. Scale-free networks are realized only at $\Delta'=0$.

VI. ''CONDENSATION'' OF LINKS ON THE ''FITTEST'' NODES

Now we can consider effects of the *s*-dependent fitness of nodes, $G=G(s)$. We proceed with our tactics, and demonstrate these effects using minimal models.

It is worth starting this section from the following notion. All networks that we study in the present paper have a general feature—*each* of their nodes has a chance to obtain a new link. Only one circumstance prevents their enrichment: seizure of this link by another node. In such kinetics of the distribution of links, there is no any finite radius of ''interaction'' and there are no principal obstacles to the capture of a great fraction of links by some node.

A. Varying fitness of nodes

Let us start from the case of a power-law dependence of the fitness, $G(s)\propto s^{-\Delta}$. At first sight, this problem seems be very close to the one considered in Sec. V. Nevertheless, a repetition of the calculations of Sec. V for the present model leads immediately to the following expression for the degree:

$$\kappa(\xi)=B\exp[\beta\xi^{-\Delta}], \tag{44}$$

where B is a constant, and $\xi=s/t$, $\overline{k}=\kappa(\xi)$ [compare with Eq. (38)]. This expression provides a value of $\overline{k}(s,t)$ different from the average degree considered in Sec. V.

If $\Delta<0$, i.e., new nodes are ''stronger'' than old ones, we obtain $\kappa(s/t\to 0)\to 0$, and the network is *exponential*. If $\Delta=0$, we obtain an ordinary scale-free network. If $\Delta>0$, then $\kappa(s/t)$ is an extremely divergent function at $\xi=0$. This peculiarity cannot be integrated. It indicates that the several oldest nodes capture a finite part of the nodes. In this case, our theory is not directly applicable, since one cannot solve a self-consistency equation and find β. We do not consider this situation in detail here. Such a behavior may be compared with the one found for $G=G(k)=k^{y-1}$, $y>1$, where most of the links turn out to be attached to the oldest node [17,38]. The ''phase diagram'' of the model is shown in Fig. 5.

B. Local strong nodes

The fluctuating fitness of nodes $G(s)$ was introduced in the recent papers a Bianconi and Barabási [23,37]. In analogy with Bose-Einstein condensation, they indicated a new phenomenon arising in this situation. A finite fraction of links may ''condense'' on a single node with the highest fitness when the distribution of the fitness of nodes has a suitable form. In Ref. [23], nodes were interpreted as energy levels which may be filled by arbitrary numbers of links, Bose particles, and the process was called Bose-Eistein condensation. However, we demonstrate the essence of the evolution of such systems using the most simple (although quite general) example, without implementation of any analogy.

Let us start from the model with directed links introduced in Sec. IV B. To simplify the formulas, we assume that $A=0$ (one may see that this does not reduce the extent of the generality of the model, which produces scaling exponents in a wide ranges of values $2<\gamma<\infty$ and $0<\beta<1$). Let us recall the model [see Fig. 1(a)]. At each increment of time, a new node with n links attached to it is added to the network. These n links are directed to the new node. Simultaneously, m extra links are distributed preferentially among nodes. The rule of preference is the same as in Sec. IV B, i.e., the probability that a link is directed to a node s is proportional to its in-degree, q_s, but with one exception—one node $\tilde{s}$ is ''stronger'' than others, i.e., the probability that this node attracts a preferentially distributed link is higher. This has an additional weight factor, fitness $g>1$, and is proportional to $gq_{\tilde{s}}$. This means that $G_s=1+(g-1)\delta_{s,\tilde{s}}$.

Let us study the behavior of the network at $t\gg\tilde{s}$. The equations for the average in-degree are

$$\frac{\partial\overline{q_{\tilde{s}}}(t)}{\partial t}=m\frac{g\overline{q_{\tilde{s}}}(t)}{(g-1)\overline{q_{\tilde{s}}}(t)+\int_0^t ds\,\overline{q}(s,t)},\quad \overline{q_{\tilde{s}}}(t=\tilde{s})=q_i,$$

$$\frac{\partial\overline{q}(s,t)}{\partial t}=m\frac{\overline{q}(s,t)}{(g-1)\overline{q_{\tilde{s}}}(t)+\int_0^t ds\,\overline{q}(s,t)},\quad \overline{q}(t,t)=n. \tag{45}$$

In the second line of Eq. (45), $s\neq\tilde{s}$. Applying Σ_s to Eq. (45) one obtains $\int_0^t ds\,\overline{q}(s,t)=(m+n)t+\mathcal{O}(1)$.

Two different situations are possible. In the first one, $q_{\tilde{s}}(t)$ grows more slowly than t, and, at long times, the denominators are equal $(m+n)t$. Then the second line of Eq. (45) is similar to Eq. (22), and we again obtain the exponents $\beta=m/(m+n)\equiv\beta_0$ and $\gamma=2+n/m\equiv\gamma_0=1+1/\beta_0$, where $0<\beta_0<1$, $2<\gamma_0<\infty$. It is convenient to write equations using not the parameters m and n, but γ_0 or β_0, i.e., the exponents of the network in which all nodes have equal fitness: $g=1$. The first line of Eq. (45), in this case, looks like

$$\frac{\partial\overline{q_{\tilde{s}}}(t)}{\partial t}=\frac{gm}{m+n}\,\frac{g\overline{q_{\tilde{s}}}(t)}{t}. \tag{46}$$

Hence, at long times, $\overline{q_{\tilde{s}}}(t)=\text{const}(q_i)t^{gm/(m+n)}$, and we see that the in-degree of the strong node grows slower than t only for

$$g<g_c\equiv 1+\frac{n}{m}=\gamma_0-1=\beta_0^{-1}>1, \tag{47}$$

so the natural threshold arises.

In the other situation, $g>g_c$, at long times, we have the only possibility: $q_{\tilde{s}}(t)=dt$, where d is some constant, since a faster growth of $q_{\tilde{s}}(t)$ is impossible. Obviously, $d<m+n$. This means that, for $g>g_c$, a finite fraction of all preferentially distributed links is captured by the strong node, and ''condensed'' on it (in Ref. [23], this situation was called Bose-Einstein condensation). We see that a single strong node may produce a macroscopic effect. In this case, Eq. (45) takes the forms

$$\frac{\partial \overline{q_{\tilde{s}}}(t)}{\partial t}=\frac{gm}{(g-1)d+m+n}\,\frac{\overline{q_{\tilde{s}}}(t)}{t},$$
$$\frac{\partial \overline{q}(s,t)}{\partial t}=\frac{m}{(g-1)d+m+n}\,\frac{\overline{q}(s,t)}{t}, \tag{48}$$

where in the second of Eqs. (48), $s\neq\tilde{s}$. Note that the coefficient in the first equation is always larger than the second, since $g_c>1$. From the first of Eqs. (48), we obtain the condition

$$\frac{gm}{(g-1)d+m+n}=1. \tag{49}$$

Therefore, above the threshold ($g>g_c$) the following fraction of all links of the network is captured by the strongest node:

$$\frac{d}{m+n}=\frac{d}{m}\,\frac{m}{m+n}=\frac{1}{g_c}\,\frac{g-g_c}{g-1}. \tag{50}$$

We have to emphasize that d is independent of the initial conditions. (Recall that we consider the long time limit.) This condensation of links leads to a change of exponents. Using Eq. (49), we immediately obtain the following expressions for them:

$$\beta=\frac{1}{g}<\beta_0, \quad \gamma=1+g>\gamma_0. \tag{51}$$

The fraction of all links captured by the strongest node, and the β and γ exponents vs g, are shown in Fig. 6. Note that the growth of g increases the value of the γ exponent. If the world were to be captured by Bill Gates or some czar, the distribution of wealth would become more fair. One should note that the strong node does not take links away from other nodes, but only *intercepts* them. The closer γ_0 is to 2, the smaller the value of g necessary to exceed the threshold. Above the threshold, values of the exponents are determined only by g. Nevertheless, the expression for $d/(m+n)$ contains γ_0 or β_0.

For $g>g_c$, in the link condensation regime, the strongest node determines the evolution of the network. With increasing time, a gap grows between the in-degree of the strongest node and the maximal in-degree of all others; see Fig. 7. A small peak at the end of the continuous part of the distribution is a trace of initial conditions; see Sec. VIII B. Note that the network remains scale free even above the threshold, i.e., for $g>g_c$, although γ grows with growing g.

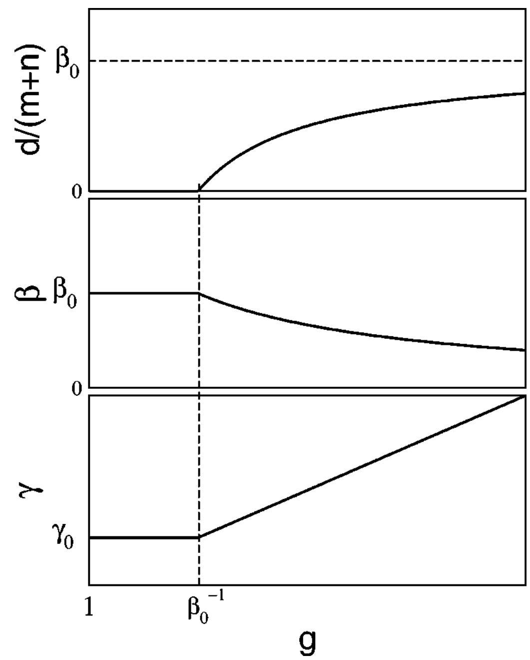

FIG. 6. ''Condensation'' of links. Shown are the fraction of all links, $d/(m+n)$, captured by a single strong node at long times, and the scaling exponents β and γ vs the relative fitness g of the strong node. The condensation occurs above the threshold value $g_c=1/\beta_0=\gamma_0-1>1$. Here β_0 and γ_0 are the corresponding exponents for a network without a strong node. $d/(m+n)[g\to\infty]\to\beta_0$ and $\beta[g\to\infty]\to 0$.

The same result may be obtained for several nodes $\tilde{s}_i$ with different values of fitness g_i, where $\tilde{s}_i\ll t$. Here $g_i>1$ (values $g_i<1$ do not produce any visible effect). In this case, the strongest node, $\tilde{s}_j$, again captures a finite part of the links if $g_j>g_c$. Note that the time required by the strongest node to attract a finite fraction of links may be very long if it is only a bit stronger than the previous strongest one. Note also that

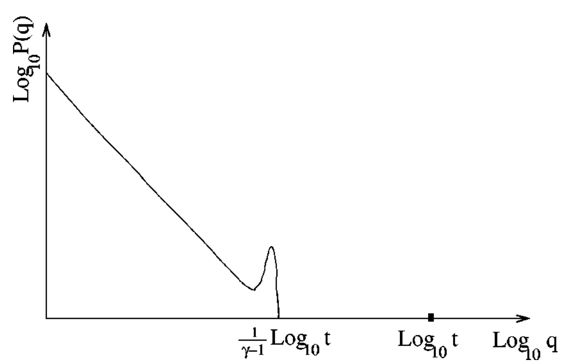

FIG. 7. Schematic plot of the degree distribution of the network with one node, the fitness of which exceeds the threshold value. The point is due to links ''condensed'' on the strong node. A hump at the cutoff of the continuous part of the distribution is a trace of the initial conditions (see Sec. VIII B and Ref. [29]).

death of the strongest node produces a dramatic effect—after some time, the distribution becomes less ''fair.'' Deletion of a single strong node may also destroy the entire network.

If N strong nodes of equal fitness g are present, one can find again that the critical exponents are described by Eq. (51). The part of the links captured by all these strong nodes together is given by Eq. (50).

Here we study the limit of large networks. One can describe relaxation to this state. Let us consider the relaxation of the fraction of all links captured by a single strong node, $\overline{q_{\tilde{s}}}(t)/[(m+n)t]$, to the final stationary value $d/(m+n)$ [Eq. (50)]. At long times, from the first of Eqs. (45), after linearization near the stationary value [Eq. (50)], we obtain the result

$$\frac{\overline{q_{\tilde{s}}}(t)-dt}{(m+n)t} \propto t^{-(g-g_c)/g}, \tag{52}$$

which is valid for any $g>g_c$. Therefore, in the regime of link condensation, the fraction of all links captured by the strong node relaxes to the final value by a power law. The exponent of this, $(g-g_c)/g$, approaches zero at the point of condensation, $g=g_c$. This behavior evokes associations with critical relaxation.

C. Mixture of nodes with different fitness

How can we ''smear'' the threshold described in Sec. V? Again, let us use the simplest model, in which nodes have two values of fitness, 1 and $g>1$; however, now the nodes with different values of fitness are distributed homogeneously. The probability that a node has a fitness equal to 1 is $1-p$; with probability p, a node has a fitness g. All other conditions are the same as in Sec. VI B. We can introduce average in-degrees $\overline{q}_1(t)$ and $\overline{q}_g(t)$ for these components. Then the evolution of the average in-degrees is determined by the equations

$$\begin{aligned}\frac{\partial \overline{q}_g(s,t)}{\partial t} &= m\frac{g\overline{q}_g(s,t)}{(1-p)\int_0^t ds\overline{q}_1(s,t)+gp\int_0^t ds\overline{q}_g(s,t)},\\ \frac{\partial \overline{q}_1(s,t)}{\partial t} &= m\frac{\overline{q}_1(s,t)}{(1-p)\int_0^t ds\overline{q}_1(s,t)+gp\int_0^t ds\overline{q}_g(s,t)},\end{aligned} \tag{53}$$

where $\overline{q}_g(t,t)=\overline{q}_1(t,t)=n$ [compare with Eqs. (45)]. As usual, we obtain

$$(1-p)\int_0^t ds\overline{q}_1(s,t)+p\int_0^t ds\overline{q}_g(s,t)=(m+n)t. \tag{54}$$

If one introduces the natural notation $m/[(1-p)\int_0^t ds\overline{q}_1(s,t)+gp\int_0^t ds\overline{q}_g(s,t)] \equiv \beta_1$, Eqs. (53) take the following forms:

$$\begin{aligned}\frac{\partial \overline{q}_g(s,t)}{\partial t} &= \beta_1 g\frac{\overline{q}_g(s,t)}{t},\\ \frac{\partial \overline{q}_1(s,t)}{\partial t} &= \beta_1\frac{\overline{q}_1(s,t)}{t}.\end{aligned} \tag{55}$$

Inserting the solutions of these equations, $\overline{q}_g(s,t)=n(s/t)^{-\beta_1 g}$ and $\overline{q}_1(s,t)=n(s/t)^{-\beta_1}$, into Eq. (54), we obtain the equation for β_1:

$$\frac{1-p}{1-\beta_1}+\frac{p}{1-\beta_1 g}=\frac{1}{1-\beta_0}. \tag{56}$$

The solution of Eq. (56) is of the form

$$\begin{aligned}\beta_1=\frac{1}{2g}\{&[1-p+p\beta_0]+[p+(1-p)\beta_0]g\\ &-\sqrt{\{[1-p+p\beta_0]+[p+(1-p)\beta_0]g\}^2-4\beta_0 g}\}\end{aligned} \tag{57}$$

(we choose this root of the square equation to obtain the proper equality, $\beta_1(g=1)=\beta_0$).

The fraction of links captured by the strong component is given by the relation

$$\frac{d}{m+n}=\frac{p}{1-\beta_1 g}\,\frac{n}{m+n}=p\,\frac{1-\beta_0}{1-\beta_1 g} \tag{58}$$

where β_1 is taken from Eq. (57). Exponents β_1 and $\gamma_1=1+1/\beta_1$ describe the distribution of nodes of the weak component containing the $1-d/(m+n)$ part of the nodes. Exponents $\beta_g=\beta_1 g$ and $\gamma_g=1+1/\beta_g$ describe the distribution of nodes of the strong component. The dependences of these characteristics on g are shown schematically in Fig. 8. One may see that, for $p\to 0$, these curves tend toward the dependences obtained for a single strong node in Sec. V. For $p>0$, a long tail of the distribution with the exponent $\gamma_g<\gamma_0<\gamma_1$ is determined by the strong nodes. For $p\to 0$, it is transformed into the δ function obtained in Sec. V. A particular value of p determines the smearing of the threshold discussed in Sec. VI B.

Thus, for $p\neq 0$, the link condensation phenomenon is absent. It exists only if there are a few nodes with maximal strength. In the case of continuous distributions $P(g)$, this happens only for special forms of $P(g)$ [23].

VII. DISTRIBUTIONS OF LINKS BETWEEN PAIRS OF NODES

Let us discuss another characteristic describing the structure of network and the distribution of links between pairs of nodes—an average matrix element of an adjacency matrix. (An element b_{ij} of an adjacency matrix is equal to 1 if there is a link connecting sites i and j, and is equals to 0 is the link is absent.)

There is a very important particular case when this characteristic may be easily calculated. In this case, new links

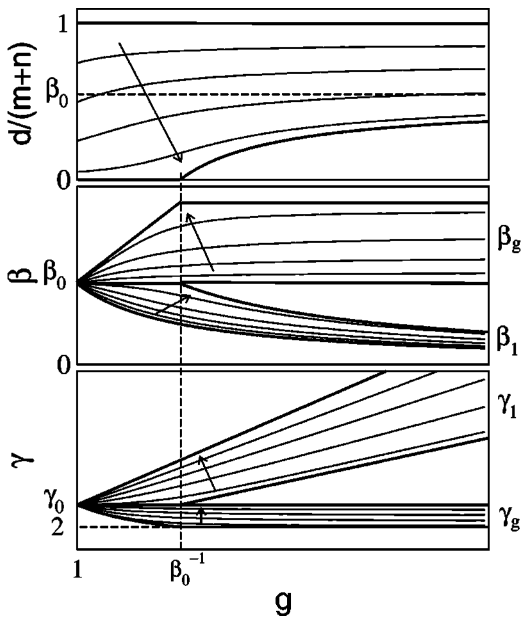

FIG. 8. Fraction of all links, $d/(m+n)$, captured by the component of the network consisting of ''strong'' nodes at long times, and the scaling exponents β and γ vs the relative fitness g of the ''strong'' nodes. We introduce two sets of exponents for two components of the network: β_1 and γ_1 for the component consisting of nodes with unit fitness [contains $(1-p)t$ nodes], and β_g and γ_g for the component consisting of nodes with a fitness g (contains pt nodes). Thin lines depict the dependences at fixed values of p. Arrows show how these curves change when p decreases from 1 to 0. At $p\to 0$, we obtain the dependences shown in Fig. 8. At $p\to 1$, $d/(m+n)\to 1$, $\beta_g(g<g_c)\to\beta_0 g$, $\beta_g(g>g_c)\to 1$, $\beta_1\to\beta_0/g$, $\gamma_1\to 1+g/\beta_0$, $\gamma_g(g<g_c)\to 1+1/(\beta_0 g)$, and $\gamma_g(g>g_c)\to\gamma_0$.

appear only between the new and old nodes, but never between old nodes of the network [see Fig. 1(b)]. For instance, networks of scientific citations belong to the class of such networks.

Here we study the simplest model of Sec. IV B with $n=0$. For brevity, in this section we consider networks with undirected links. Then into the relations of Sec. IV B, we can substitute $\overline{q}(s,t)=\overline{k}(s,t)-m$. That is, at each time step, we add a new node with m attached undirected links. Other ends of these links are distributed preferentially between old nodes.

Let us introduce an average number of links between nodes s and s' ($s'>s$) at a time t, $\overline{b}(s,s',t)$. For the network that we consider,

$$\overline{k}(s,t)=\int_0^s du\,\overline{b}(u,s,t)+\int_s^t du\,\overline{b}(s,u,t)$$
$$=m+\int_s^t du\,\overline{b}(s,u,t). \tag{59}$$

No new links between old nodes arise, so one can see that

$$\overline{b}(s,t,t'\geq t)=\overline{b}(s,t,t),\qquad \frac{\partial\overline{k}(s,t)}{\partial t}=\overline{b}(s,t,t'\geq t). \tag{60}$$

Using a scaling representation, $\overline{k}(s,t)=\kappa(s/t)$ and $\overline{b}(s,s',t)=t^{-1}\mathcal{B}(s/t,s'/t)$, one can write

$$\overline{b}(s,t,t'\geq t)=\frac{1}{t'}\mathcal{B}\left(\frac{s}{t'},\frac{t}{t'}\right)=\frac{1}{t}\mathcal{B}\left(\frac{s}{t},\frac{t}{t}=1\right) \tag{61}$$

and

$$\frac{\partial}{\partial(s/t)}\kappa\left(\frac{s}{t}\right)=\frac{\partial}{\partial(s/t)}\kappa\left(\frac{s}{t'}\,\frac{t'}{t}\right). \tag{62}$$

Therefore, we obtain relations for $\mathcal{B}(\xi,\xi')$ and $\kappa(\xi)$,

$$\mathcal{B}(\xi,\xi')=\frac{1}{\xi'}\mathcal{B}\left(\frac{\xi}{\xi'},1\right),$$
$$\mathcal{B}(\xi,\xi')=-\frac{\xi}{\xi'^2}\kappa'\left(\frac{\xi}{\xi'}\right), \tag{63}$$
$$\mathcal{B}(\xi,1)=-\xi\kappa'(\xi)$$

where $\kappa'(x)\equiv d\kappa(x)/dx$.

In our particular problem, $\kappa(1)=m$; thus, from Eq. (59), it follows that

$$\kappa(\xi)=m+\int_\xi^1 d\xi''\mathcal{B}(\xi,\xi'')=m+\int_\xi^1\frac{d\zeta}{\zeta}\mathcal{B}(\zeta,1). \tag{64}$$

From relations of Sec. IV B, we obtain

$$\kappa(\xi)=m\left[2-\frac{1}{\beta}+\left(\frac{1}{\beta}-1\right)\xi^{-\beta}\right]. \tag{65}$$

Then, from Eq. (63), we obtain the result,

$$\mathcal{B}(\xi,\xi')=m(1-\beta)\xi^{-\beta}\xi'^{\beta-1}. \tag{66}$$

This characteristic was obtained explicitly for a model with $\beta=1/2$ [29]. From this exact solution, the particular case $\beta=1/2$ of Eq. (66) follows immediately. Note that $\mathcal{B}(\xi,\xi')$ is not proportional to the product $\kappa(\xi)\kappa(\xi')$.

It is interesting to compare these expressions with corresponding results for exponential networks (we shall discuss the applicability of the continuous approach to non-scale-free networks in Sec. X C). Setting $m=n_r$ and $n=A=0$ in the equations of Sec. IV C, one obtains

$$\kappa(\xi)=m(1-\ln\xi),\qquad \mathcal{B}(\xi,\xi')=\frac{m}{\xi'}. \tag{67}$$

Equation (67) follows from Eqs. (65) and (66) at $\beta\to 0$.

We have obtained the density of linkage between pairs of nodes with fixed times of birth. From this characteristic (in the continuous approximation), we can find an average number of connections $\mathcal{D}(k,k')$ of parent nodes with degree k, and child nodes with degree k'. In the continuous approximation, $k>k'$.

We consider only the scaling region, and will not write coefficients in the following formulas of the present section. $k=\kappa(\xi)$. Then, for a scale-free network, $P(k)=-(\partial\kappa/\partial\xi)^{-1}\sim k^{-1/\beta-1}$. Analogously,

$$\mathcal{D}(k,k')=B(\xi,\xi')\left(\frac{\partial k}{\partial\xi}\right)^{-1}\left(\frac{\partial k'}{\partial\xi'}\right)^{-1}\;[\xi=k^{-1/\beta},\xi'=k'^{-1/\beta}]$$
$$\propto k^{-1/\beta}k'^{-2}. \quad (68)$$

One sees that $\mathcal{D}(k,k')$ is a decreasing function of both k and k'. Therefore, if k (the parent node) is fixed, the most probable linking is with a child node with the smallest possible k', i.e., a node with $k'\sim 1$.

If k' (the degree of a child node) is fixed, the most probable linking is with a parent node with the smallest possible k, i.e., a node with $k=k'$. This answer may be compared with the corresponding exact result of Krapivsky and Redner [38]—the maximum probability that a node with degree k (parent) and a node with degree k' (child) are connected, when k' is fixed, occurs at $k/k'=0.372$. Therefore, we see that the continuous approach is also good in this situation.

Equation (68) appears asymmetrical, and cannot be factorized to the product $P(k)P(k')$. The reason for this is the obvious absence of the symmetry relatively reversal of time—a quite natural asymmetry between parents and children.

VIII. ACCELERATING GROWTH

Above we studied only linear network growth, i.e., the total number of links of a network divided by the total number of its nodes was constant during the evolution. This is only a particular case of network growth. The total number of links may be a nonlinear function of the total number of nodes. Keeping in mind the most intriguing applications, communications networks, we now concentrate on accelerating growth [36], where this function grows more rapidly than a linear one. In particular, we will see that a power-law dependence of the input flow of links produces scale-free networks.

A. Scaling relations

In the present section, we consider scale-free networks with an input flow of links that depends on t by a power law. We will see that, in the limit of a large network size, such nonlinear growth may produce the nonstationary distribution

$$P(q,t)\propto t^{z}q^{-\gamma}, \quad (69)$$

and average in-degree (or degree)

$$\overline{q}(s,t)\propto t^{\delta}\left(\frac{s}{t}\right)^{-\beta}. \quad (70)$$

Let us discuss the result of such a nonstationary behavior.

One can easily repeat the derivation of Sec. III, and obtain the scaling form of the degree distribution of individual nodes:

$$p(q,s,t)=t^{-\delta}\left(\frac{s}{t}\right)^{\beta}f\left[qt^{-\delta}\left(\frac{s}{t}\right)^{\beta}\right]. \quad (71)$$

We again use the definition of $P(k,t)$ [Eq. (7)] and Eq. (71):

$$\int_0^\infty dx\,t^{-\delta}x^{\beta}f(qt^{-\delta}x^{\beta})\propto t^{\delta/\beta}q^{-1-1/\beta}\propto t^{z}q^{-\gamma}. \quad (72)$$

Hence we obtain a universal scaling relation for the exponents z, δ, and β,

$$z=\delta/\beta, \quad (73)$$

and the old relation [Eq. (12)] for the exponents γ and β.

Let us derive this result once again, applying the continuous approach. Using Eqs. (11), (69), and (70), we obtain

$$t^{z}q^{-\gamma}\propto -\frac{1}{t}\frac{\partial s}{\partial t}[q=t^{\delta+\beta}s^{-\beta}]\propto t^{\delta/\beta}q^{-1-1/\beta}. \quad (74)$$

From Eq. (74), relations (12) and (73) follow immediately.

One can write Eq. (71) using only the γ exponent:

$$p(q,s,t)=\frac{s^{1/(\gamma-1)}}{t^{(1+z)/(\gamma-1)}}f\left(q\frac{s^{1/(\gamma-1)}}{t^{(1+z)/(\gamma-1)}}\right). \quad (75)$$

Equation (75) is a direct generalization of Eq. (14), obtained for linearly growing networks.

B. Cutoff of degree distribution

Equation (69) is valid only in the limit of large network size (long times). Let us briefly discuss the finite-size effects that arise.

Relation (70) of Sec. VIII A, together with a rapid decrease of $p(q,s,t)$ at large q, produces a cutoff of the power-law distribution at the characteristic value

$$q_{cut}\sim t^{\beta+\delta}=t^{\beta(1+z)}=t^{(1+z)/(\gamma-1)}. \quad (76)$$

Here we use the fact that $\overline{q}(s,t)$ is largest for the oldest node and the scaling relations between the exponents [Eqs. (12) and (73)]. The cutoff, q_{cut} can be also obtained from the estimation $t\int_{q_{cut}}^{\infty}dq\,t^{z}q^{-\gamma}\sim 1$. It was shown that a trace of the initial conditions at $q\sim q_{cut}$ may be visible in degree and in-degree distributions measured for any network sizes [29]. Such a cutoff (and a trace of the initial conditions) imposes restrictions on the observation of power-law distributions, since there are few huge networks in Nature.

Let us obtain a general form of $P(q,t)$ for scale-free networks in a scaling regime. Using the known scaling form of $p(q,s,t)$ [Eq. (75),] we can write

$$P(q,t)=\frac{1}{t}\int_1^t ds\,p(q,s,t)$$
$$\sim t^{z}q^{-\gamma}\int_{qt^{-(1+z)/(\gamma-1)}}^{qt^{-z/(\gamma-1)}}dw\,w^{\gamma-1}f(w). \quad (77)$$

Passing to the scaling limit, $q\to\infty$, $t\to\infty$, and fixed $qt^{-(1+z)/(\gamma-1)}$, we can replace the upper limit of the integral in Eq. (77) by infinity. Then we immediately obtain the scaling form

$$P(q,t)=t^z q^{-\gamma}F(qt^{-(1+z)\beta})=t^z q^{-\gamma}F(qt^{-(1+z)/(\gamma-1)}), \tag{78}$$

where $F(w)$ is a scaling function. In the case of a linearly growing network, the exponent z is zero, so $P(q,t)=q^{-\gamma}F(qt^{-1/\beta})=q^{-\gamma}F(qt^{-1/(\gamma-1)})$. This relation was obtained for an exactly solvable model [29].

C. Scaling exponents

Using relations obtained in Secs. VIII A and VIII B one can obtain general results for a nonlinear, accelerating growth of networks. We start from the most general considerations. In scale-free networks, a wide range of the degree distribution function is of a power-law form, $P(q)\propto t^z q^{-\gamma}$. It will be clear from the following that, to keep a network in the class of free-scale nets, the flow of new links has to be a power function of the number of nodes of the network, i.e., be proportional to t^a. Here we introduce a new exponent a.

First let us assume that the exponent of the distribution is less than 2. The reasonable range is $1<\gamma<2$. To produce the restricted average degree (that is, proportional to t^a), the distribution must have a cutoff at large q, $q_c\sim t^{(1+z)/(\gamma-1)}$ (see Sec. VIII B). $P(q)\propto t^z q^{-\gamma}$ for $q_l\sim t^x\lesssim q\lesssim q_c$. The restriction from below is necessary to guarantee convergence of the integral, $\int_0^\infty dqP(q,t)=1$. From this, we immediately obtain $x=z/(\gamma-1)$. (Of course, this relation is also valid for $\gamma>2$.)

The average degree $\overline{q}$ is of the order t^{a+1}/t, then $t^a\sim\int^{t^{(1+z)/(\gamma-1)}}dq\,qt^z q^{-\gamma}\sim t^{-1+(1+z)/(\gamma-1)}$ (the value of the integral is determined by its upper limit). Therefore, $(1+z)/(\gamma-1)=a+1$, so the cutoff of the distribution is of the order of the total number of links in the network. This is the maximal number of the problem; hence the cutoff is absent. The expression for the γ exponent, $\gamma=1+(1+z)/(1+a)$, follows from the last relation.

Note that a is an ''external'' exponent which governs the growth process. Hence we have demonstrated that it is sufficient to know a and only one exponent of γ, β, z, δ, or x for finding all the others.

Also note that we have to set $z<a$ to keep the exponent γ below 2, as assumed. Also, one sees that the lower boundary for γ, $1+1/(1+a)$, is approached for the stationary distribution $z=0$. In this case, the form of the distribution is completely fixed by the accelerating growth, the exponent γ depends only on a.

The other possibility is $\gamma>2$. In this case, the integral for the average degree is determined by its lower limit $t^a\sim\int_{t^{z/(\gamma-1)}}dq\,qt^z q^{-\gamma}\sim t^{z-z(\gamma-2)/(\gamma-1)}$. Hence $\gamma=1+z/a$ and $z>a$. (Of course, this relation is not valid for $a=0$.) Thus we have described the possible forms of the degree distribution.

Let us demonstrate how these distributions may arise in nonlinearly growing networks with preferential linking. We introduce the simplest generalizations of the model of Sec. IV B to the case of an increasing input flow of links, c_0t^a and $a>0$.

First let us consider the case of constant additional attractiveness, $A=$const. The equation for $\overline{q}(s,t)$ is

$$\frac{\partial\overline{q}(s,t)}{\partial t}=c_0t^a\frac{\overline{q}(s,t)+A}{\int_0^t du[\overline{q}(u,t)+A]}, \tag{79}$$

where $\overline{q}(0,0)=0$, $\overline{q}(t,t)=n$. One may check that $\int_0^t du\overline{q}(u,t)=nt+c_0t^{a+1}/(a+1)$. Inserting this relation into Eq. (79), and solving the resulting equation, one obtains

$$\frac{\overline{q}(s,t)+A}{n+A}=\left[\frac{1+(n+A)(1+a)t^{-a}/c_0}{1+(n+A)(1+a)s^{-a}/c_0}\right]^{1+1/a}\left(\frac{s}{t}\right)^{-(a+1)}. \tag{80}$$

In the interval $[(n+A)(1+a)/c_0]^{1/a}\ll s\ll t$,

$$\overline{q}(s,t)=(n+A)\left(\frac{s}{t}\right)^{-(a+1)}. \tag{81}$$

Thus the exponent β, $\overline{q}(s,t)\propto s^{-\beta}$ is equal to $1+a$, and is larger than 1. The dependence $\overline{q}(s)$ becomes constant,

$$\overline{q}(s,t)=(n+A)^{-1/a}\left(\frac{c_0}{1+a}\right)^{1+1/a}t^{a+1}, \tag{82}$$

at $s\ll[(n+A)(1+a)/c_0]^{1/a}$. One may compare the result [Eq. (82)] with the total number of links in the network, $N(t)\approx c_0t^{a+1}/(1+a)$.

From Eq. (82), we see immediately that the exponent $\beta=1+a$, so $\gamma=1+1/a$ and $\delta=z=0$. One may calculate the degree distribution using Eq. (11). The resulting distribution, in the region $1\ll q/(n+A)\ll\{c_0/[(n+A)(1+a)]\}^{1+1/a}t^{1+a}$, is of the form

$$P(q,t)=\frac{(n+A)^{1/(1+a)}}{1+a}q^{-[1+1/(1+a)]}. \tag{83}$$

Thus we obtain the stationary degree distribution with a γ exponent less than 2 that belongs to one of the types described above.

To demonstrate the other possibility, $\gamma>2$, below we consider the model with a different rule of the distribution of new links. Let the additional attractiveness be time dependent, and new links be distributed between nodes with probability proportional to $q+Bc_0t^a/(1+a)$, where B is positive constant. $c_0t^a/(1+a)$ is the average degree of the network at time t.

Repeating the previous calculations, one obtains the equation

$$\frac{\partial\overline{q}(s,t)}{\partial t}=c_0t^a\frac{\overline{q}(s,t)+Bc_0t^a/(1+a)}{nt+Bc_0t^{a+1}/(1+a)+c_0t^{a+1}/(a+1)}, \tag{84}$$

where $\overline{q}(0.0)=0$, and $\overline{q}(t,t)=n$. At long times, one obtains

$$\frac{\partial\overline{q}(s,t)}{\partial t}=\frac{1+a}{1+B}\,\frac{\overline{q}(s,t)+Bc_0t^a/(1+a)}{t}. \tag{85}$$

The solution of Eq. (85) is

$$\overline{q}(s,t)=\left[n+\frac{Bc_0s^a}{1-Ba}\right]\left(\frac{s}{t}\right)^{-(1+a)/(1+B)}-\frac{Bc_0t^a}{1-Ba}. \tag{86}$$

If $B=0$, we obtain the previous result, $\beta=1+a$. For $s^a \gg n(1-Ba)/(Bc_0)$,

$$\overline{q}(s,t)\approx\frac{Bc_0t^a}{1-Ba}\left\{\left(\frac{s}{t}\right)^{a-(1+a)/(1+B)}-1\right\}. \tag{87}$$

Therefore, the scaling exponents of the growing network are $\beta=(1+a)/(1+B)-a=(1-Ba)/(1+B)$, $\gamma=1+1/\beta=1+[(1+a)/(1+B)-a]^{-1}=2+B(1+a)/(1-Ba)$, $\delta=a$, and $z=a(1+B)/(1-Ba)$. The degree distribution differs sharply from the distribution obtained for the previous model. It is nonstationary, and is of the form $P(q,t)\sim t^{a(1+B)/(1-Ba)}q^{-[1+(1+B)/(1-Ba)]}$ for $q\gg t^a$. In this case, $\beta<1$ and $\gamma>2$ for any positive a and B. The scaling regime is realized if $Ba<1$. Note that, in both cases considered, one cannot set $a=0$ directly in the obtained expression for the scaling exponents.

IX. DEVELOPING AND DECAYING NETWORKS

Now we can study the evolution of the network accompanied by a reconstruction of its old part [35]. This, e.g., may include a permanent deletion of old links or nodes. Note that the processes of addition and deletion of links may be considered in a unified way, so we study them together.

A. Developing networks

Let us introduce two channels for the appearance of new links. The first one was studied in Sec. IV B. We consider undirected links, so, in our old formulas, we have to substitute $\overline{q}(s,t)=\overline{k}(s,t)-m$ and put $\overline{k}(t,t)=m$. $n=0$. Instead of the additional attractiveness A, here we use the constant $A_n=A-m>-m$. The second channel is the following. Each time a new node is added, c additional links arise between old unconnected nodes i and j, with a probability proportional to the product $(k_i+A_o)(k_j+A_o)$, $A_o>-m$. Note that A_n and A_o may be not equal; this here we have a mixture of different preferences as in Sec. IV C. Then the equation for $\overline{k}(s,t)$ is of the form

$$\frac{\partial\overline{k}(s,t)}{\partial t}=m\frac{\overline{k}(s,t)+A_n}{\int_0^t du[\overline{k}(u,t)+A_n]}+2c\frac{\overline{k}(s,t)+A_o}{\int_0^t du[\overline{k}(u,t)+A_o]} \tag{88}$$

(see Ref. [35]). $\int_0^t ds\overline{k}(s,t)=2(m+c)t$, so we obtain immediately the scaling exponent β,

$$\beta=\frac{m}{2m+2c+A_n}+\frac{2c}{2m+2c+A_o}, \tag{89}$$

and, using Eq. (12), the exponent γ. We do not consider these relations in detail.

B. Decaying networks

One can consider the possibility of deletion of links. In this case, the second channel is the following. At each increment of time, $-c$ random links are removed from the network. Note that we keep the same definition of c as in Sec. IX A, in order to use similar relations.

Previously, a process of instant random damage was considered [16,25–28]. Such a type of damage cannot change the value of γ either for random removal links or for removal nodes. Here we consider quite a different situation, *permanent random damage*, in which components of a network are removed permanently during its growth. In this case, the difference between random removal of nodes and links is striking.

We have shown that Eqs. (88) and (89) also describe the case of decaying networks with a permanent deletion of links, if one takes c to be negative in them and assumes that $A_o=0$ [35]. Therefore, Eq. (89), and the corresponding relation for γ, provide the exponents in this situation. These results have been checked by simulation in the particular case of $A_n=0$, i.e., $\gamma(c=0)=3$ [35].

Here we present the corresponding relations for in-degree exponents network with a permanent deletion of directed links. As a base, we use the model with a preferential attachment of links, introduced in Sec. IV B but, with $n=0$. One can easily check that

$$\beta=\frac{1}{1+c/m+A_n/m}+\frac{c/m}{1+c/m}=\frac{1+\gamma_0c/m+(c/m)^2}{(1+c/m)(\gamma_0-1+c/m)} \tag{90}$$

and

$$\gamma=2+\frac{\gamma_0-2}{1+\gamma_0c/m+(c/m)^2}, \tag{91}$$

where $\gamma_0\equiv\gamma(c=0)=2+A_n/m$. The resulting phase diagram $-c/m$ vs γ_0 is shown in Fig. 9. One sees that the random removal of links increases the γ exponent, which grows monotonically with increasing $-c/m$ until it becomes infinite on the line $\gamma_0=(-c/m)+1/(-c/m)$. In the dashed region of Fig. 9, the network is out of the class of scale-free nets. Note that, for large enough $-c/m$, the network may decay to a set of uncoupled clusters.

Let us now consider an additional permanent deletion of random nodes. *Ab initio*, one may expect that this factor does not change γ. Nevertheless, as we shall see, this case is of special interest.

At each increment of time, one new node is added, and a randomly chosen node is deleted with probability $c\leqslant 1$. We use the preferential linking introduced in Sec. IV B, although

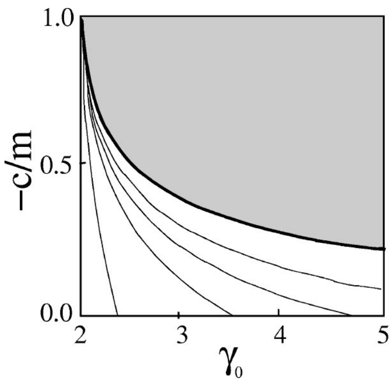

FIG. 9. Phase diagram of a network growing under the condition of permanent random damage—permanent deleting of random directed links. At each time step, m new links are added and $-c$ random links are deleted (see the text). γ_0 is the scaling exponent of the corresponding network growing without deletion of links. Curves in the plot are lines of constant values of γ. $\gamma=\infty$ on the line, and $\gamma_0=-(c/m)-1/(c/m)$. In the dashed region, the network is out of the class of scale-free nets.

a more general model may be also considered. It is natural to consider in-degree, $q(s,t)$, here, since results are more general in this case.

Let us introduce the probability that a nodes s is present, $\mathcal{N}(s,t)$. In the continuous approach this has the meaning of the ''density'' of surviving nodes at time t, $\overline{\mathcal{N}}(s,t)$. One may introduce the ''density of in-degree.'' In the continuous approach, this looks like

$$\overline{d}(s,t)\equiv\overline{\mathcal{N}}(s,t)\overline{q}(s,t). \tag{92}$$

Our main equation for $\overline{\mathcal{N}}(s,t)$ is of the form

$$\frac{\partial\overline{\mathcal{N}}(s,t)}{\partial t}=-c\frac{\overline{\mathcal{N}}(s,t)}{\int_0^t du\overline{\mathcal{N}}(u,t)}, \tag{93}$$

where $\overline{\mathcal{N}}(0,0)=0$ and $\overline{\mathcal{N}}(t,t)=1$. From Eq. (93) one immediately obtains the obvious relation $\int_0^t du\overline{\mathcal{N}}(u,t)=(1-c)t$. The solution of Eq. (93) is

$$\overline{\mathcal{N}}(s,t)=\left(\frac{s}{t}\right)^{c/(1-c)}. \tag{94}$$

In the present case, the equation for the in-degree is

$$\frac{\partial\overline{q}(s,t)}{\partial t}=m\frac{\overline{q}(s,t)+A}{\int_0^t du\overline{\mathcal{N}}(u,t)[\overline{q}(s,t)+A]}, \tag{95}$$

where the integral on the right side is equal to the sum of the total in-degree of the network at time t and the product of the additional attractiveness and the number of survived nodes [compare with Eq. (19) for $c=0$]. Here $\overline{q}(0,0)=0$ and $\overline{q}(t,t)=n$, so $\overline{d}(0,0)=0$ and $\overline{d}(t,t)=n$.

Multiplying Eq. (95) by $\overline{\mathcal{N}}(s,t)$, and applying $\int_0^t ds$ to both sides of the resulting equation, we obtain $\int_0^t ds\overline{d}(s,t)=(m+n)(1-c)t$. Substituting this relation into Eq. (95), we immediately obtain $\overline{q}(s,t)\propto(s/t)^{-\beta}$, where the exponent is

$$\beta=\frac{m}{(1-c)(m+n+A)}=\frac{\beta_0}{1-c}. \tag{96}$$

Here $\beta_0\equiv\beta(c=0)$, $\gamma_0\equiv\gamma(c=0)$. Using Eq. (94), we obtain for the density of in-degree, $\overline{d}(s,t)\propto(s/t)^{-(\beta+c)/(1-c)}$. Note that $\overline{d}(s\to 0,t)\to\infty$ for $c<\beta_0$ and $\overline{d}(s\to 0,t)\to 0$ for $c>\beta_0$.

The expression for the degree distribution looks like

$$P(k,t)=\frac{\int_0^t ds\mathcal{N}(s,t)p(q,s,t)}{\int_0^t ds\mathcal{N}(s,t)}=\frac{\int_0^t ds\overline{\mathcal{N}}(s,t)\,\delta(q-\overline{q}(s,t))}{\int_0^t ds\overline{\mathcal{N}}(s,t)}. \tag{97}$$

Therefore, repeating the derivation of Sec. III, we obtain the distribution $P(k)\propto k^{-(1/\beta)[c/(1-c)]}k^{-1-1/\beta}\propto k^{-1-1/[\beta(1-c)]}$, so $\gamma=1+1/[\beta(1-c)]=1+1/\beta_0=\gamma_0$. Thus the distribution is of the same form as that without a permanent random deleting of nodes. Note that this situation differs sharply from the case of permanent random deletion of links considered above.

Here we find a violation of the scaling relation [Eq. (12)]. The reason for this is an effective renormalization of the s variable due to the removal of nodes. Repeating the scaling considerations of previous sections, for this case we obtain the following forms: $p(q,s,t)=(s/t)^{\beta}f[q(s/t)^{\beta}]$ and $P(q)=q^{-\{1+1/[\beta(1-c)]\}}F(q/t^{\beta})$.

We conclude this section with the statement that permanently deleting a part of the nodes with the largest values of degree (that is, in analogy to an intentional attack [25,28]), one destroys the scaling behavior of the network. One can easily check this statement by using the continuous approach.

X. APPLICABILITY OF THE CONTINUOUS APPROACH

In the present section, we discuss the quality of the continuous approach employed here. For this, we compare known exact results with the corresponding results obtained in the framework of the continuous approximation.

A. Linear preference

In Sec. II, we have already written out the answer to the continuous approach for the Barabási-Albert model with $m=1$. For a more general case, when m is any positive integer number, and $k=q+m$, in the framework of the continuous approach one obtains $P(q)=2m^2/(q+m)^3$ [19]. One may compare this expression with the exact result obtained without passing to a continuous limit: $P(q)=2m(m+1)/[(q+m)(q+m+1)(q+m+2)]$ [18]. The exponents are the same but the factors are different.

One may ask why this approach is so good. The reason is the rapid decrease of $p(q,s,t)$ at large q. Because of this, the results of the continuous approximation obtained with the δ-function ansatz [Eq. (5)] are reasonable. Indeed, in the limit of large s and t, and for fixed s/t, we obtained the scaling, exponentially decreasing, expression $p(q,s,t) = [q(s/t)^{\beta}]^{A-1}\exp[-q(s/t)^{\beta}]/\Gamma(A)$ [18]. Here $\Gamma(\ )$ is the gamma function. This relation is valid for the model of Sec. IV B, with $n=0$.

B. Nonlinear preference

The continuous approach may be applied to nonlinear preference. In this case, calculations similar to the ones of Sec. V may be made.

We consider the simplest case, generalizing the Barabási-Albert model $\overline{k}(t,t)=1, m=1$. Then the main equations are

$$\frac{\partial \overline{k}(s,t)}{\partial t} = \frac{f_p[\overline{k}(s,t)]}{\int_0^t du f_p[\overline{k}(u,t)]} \Rightarrow \int_0^t ds \overline{k}(s,t) = 2t. \qquad (98)$$

Here $f_p(k)$ is a preference function. Let us search for the solution of Eq. (98) in the scaling form $\overline{k}(s,t)=\kappa(s/t)$. Then

$$-\frac{\partial \kappa(\xi)}{\partial \ln \xi} = \frac{f_p[\kappa(\xi)]}{\int_0^1 d\zeta f_p[\kappa(\zeta)]}, \quad \kappa(1)=1, \quad \int_0^1 d\zeta \kappa(\zeta) = 2. \qquad (99)$$

We start from a specific type of nonlinear preference that produces scale-free networks. Let the probability for a distribution of new links be proportional to the preference function $f_p(k)$, that is asymptotically linear for large k, $f_p(k \to \infty) \to ck$, where c is a constant (see Refs. [17,38]).

The integral $\int_0^1 d\zeta f_p[\kappa(\zeta)]$ is a constant of the problem. In the scaling region of large $\overline{k}(s,t)$ and κ, the equation takes the form

$$-d \ln \kappa(\xi)/d \ln \xi = c\left\{\int_0^1 d\zeta f_p[\kappa(\zeta)]\right\}^{-1} = \beta. \qquad (100)$$

Equation (100) demonstrates that scaling is present, and the network is scale free in this case.

Therefore, to find the scaling exponent β, we have to solve the equation

$$-\frac{\partial \kappa(\xi)}{\partial \ln \xi} = \frac{\beta}{c} f_p[\kappa(\xi)]. \qquad (101)$$

After inserting its solution $\kappa(\xi) = F^{-1}[F(1) + (\beta/c)\ln \xi]$ $(F(\kappa) \equiv \int d\kappa f(\kappa)$, where F^{-1} is an inverse function, into

$$\beta^{-1} = c^{-1}\int_0^1 d\zeta f_p[\kappa(\zeta)]$$

or, equivalently,

$$2 = \int_0^1 d\zeta \kappa(\zeta), \qquad (102)$$

we find the solution β of any of these transcendental equations.

We will not consider examples of applications of these relations, but will briefly describe the case of a power-law preference function just to test the continuous approach using the known non-scale-free network. Indeed, we have already checked the quality of the continuous approach for scale-free networks. Now it is natural to test it for other networks.

Let the preference function be $f_p(k) = k^{-y}$. If one sets

$$\int_0^1 d\zeta \kappa^y(\zeta) = \mu = \text{const}, \qquad (103)$$

then the equation for $\kappa(\xi)$ is

$$-\frac{d \ln \kappa(\xi)}{d \ln \xi} = \mu^{-1}\exp[-(1-y)\ln \kappa]. \qquad (104)$$

Its solution is

$$\kappa(\xi) = \left(1 - \frac{1-y}{\mu}\ln \xi\right)^{1/(1-y)}. \qquad (105)$$

The constant μ can be obtained from the transcendental equations

$$2 = \int_0^1 d\xi \left(1 - \frac{1-y}{\mu}\ln \xi\right)^{1/(1-y)}, \qquad (106)$$

or, equivalently,

$$\mu = \int_0^1 d\xi \left(1 - \frac{1-y}{\mu}\ln \xi\right)^{y/(1-y)}. \qquad (107)$$

The final transcendental equation may be written in the form

$$2\left(\frac{\mu}{1-y}\right)^{1/(1-y)} e^{-\mu/(1-y)} = \Gamma\left(1 + \frac{1}{1-y}, \frac{\mu}{1-y}\right), \qquad (108)$$

which gives $\mu(y)$. Here $\Gamma()$ is the incomplete gamma function. Near $y=1$, $\mu \cong 2y$, and near $y=0$, $\mu \cong 1+0.5963y$, where $0.5963 = e\,\mathrm{Ei}(-1)$, $\mathrm{Ei}()$ is the exponential integral.

Inserting Eq. (105) into the expression for $P(k)$ in the continuous approach [Eq. (11)] we obtain the degree distribution:

$$P(k) = \mu e^{\mu/(1-y)} k^{-y} \exp\left[-\frac{\mu}{1-y}k^{1-y}\right], \qquad (109)$$

These results are close to the corresponding exact ones [17,38]. The values of the powers are the same, although the coefficients μ differ slightly. That is, in Refs. [17,38], near $y=1$, $\mu \cong 2 - 2.407(1-y)$; near $y=0$, $\mu \cong 1+0.5078y$.

C. Random attachment of links

Finally, in the framework of the continuous approach, let us consider the model discussed in Sec. IV A, producing non-scale-free, exponential degree distributions (see Refs. [5,19]). This network is a particular case of the network considered in Sec. IV C, where $n=0$, $m=0$, $n_r=1$, and $A=0$. The solution of Eq. (27) in this case is

$$\overline{k}(s,t)=1-\ln(s/t) \tag{110}$$

[we used the boundary condition $\overline{k}(t,t)=1$]. Then, using Eq. (11), one obtains the degree distribution

$$P(k)=-\frac{1}{t}\frac{\partial[t\exp(1-k)]}{\partial k}=\exp(1-k) \tag{111}$$

(also see Refs. [5,19]). This can be compared with the exact form of Sec. IV A: $P(k)=\exp(-k\ln 2)$.

Therefore, the continuous approach easily produces reasonable answers even for non-scale-free networks. The reason for this is again the rapid decrease of $p(k,s,t)$ at large k for these networks, see Eq. (18) of Sec. IV A.

XI. CONCLUSIONS

We have analyzed scale-invariant properties of scale-free networks whose growth is governed by a mechanism of preferential attachment of links. Degree distributions of such networks are of a simple scaling form. We have shown that the scaling exponents that arise are coupled by universal scaling relations. Nevertheless, we present an important particular case in which these simple relations are violated.

One of the questions discussed was about types of preferential attachment of links producing scale-free networks. One can see that scale-free networks are produced by a wide variety of linkings. In particular, it is enough to add an admixture of linear preference linking to a random attachment of links to obtain a scale-free network. The interplay of different factors, such as the deletion of links of a network during its growth, may dramatically change its degree distribution, and even remove it from the class of scale-free networks. Hence we have shown how one can change the critical exponents of a network.

At all times, we used a very simple continuous approach. Why is this so good? The scaling behavior of networks arises from a power-law singularity of degree at $s=0$; i.e., for the oldest nodes, $\overline{k}(s,t)\propto s^{-\beta}$. Thus, ''the oldest are the richest,'' and such a behavior is often perceived as a defect of a preferential linking scheme [39]. If one removes such a singularity or makes it weaker, the growing network will be out of the class of scale-free nets. It is the separation of power and exponential dependences in $P(q,t)$ and $p(q,s,t)$ that makes the continuous approximation so efficient. Therefore, we see that scale-free networks are quite suitable for the continuous approach.

Natural boundaries $s=0$ and $s=t$ are always present in growing network. We showed that, even the presence of strong nodes did not lead to a violation of the rule ''the oldest are the richest'' for nodes in the continuous part of degree distribution. Nevertheless, we have found a threshold value of the strength of this node, above which a single node influences the evolution of an entire network. This captures a finite fraction of all links—condensation of links—and determines values of exponents, although the network remains scale free. These ''collective'' effects are explained by the absence of any ''interaction distance'' in the process of network growth. Each node has some chance to obtain a new link. Therefore, in principle, it is possible to adjust parameters to direct all new links to a single node.

One should note that the main part of our paper has been devoted to the study of the one-node characteristic of the network—degree distribution. The same simple characteristic is a matter of interest to most modern experimental and theoretical studies. This restriction lets us apply typically rather general models, in which links appeared between arbitrary nodes, e.g., new and old or old and old. If we do not study the statistics of connections between different nodes and global connectivity properties of the network, the considered problems are usually equivalent to a classical problem of distribution of new particles among an increasing number of boxes. In fact, in the present paper, we studied the question of how a network is self-organized into a scale-free structure using some versions of stochastic multiplicative processes [22] which are most known by their ''econophysical'' applications. Most of our results may be described both in terms of the theory of evolving networks and econophysics—wealth distribution processes.

Nevertheless, a part of our paper (see Sec. VII) has been devoted to problems that cannot be understood using a one-node characteristic. The main goal of statistical physics of growing networks—a description of the topology of evolving networks—is located just in this direction.

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[1] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[2] B. A. Huberman and L. A. Adamic, Nature (London) **401**, 131 (1999).
[3] R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins, in *Proceedings of the 25th VLDB '99 Conference (Edinburgh, 1999)* (Morgan Kaufman, New York, 1999), pp. 639–650.
[4] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, and J. Wiener, in *Proceedings of the 9th WWW Conference (Amsterdam, 2000)* (Elsevier, Amsterdam, 2000), pp. 309–320.
[5] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[6] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comput. Commun. Rev. **29**, 251 (1999).
[7] P. Baran, *Introduction to Distributed Communications Networks*, RM-3420-PR, August 1964, http://www.rand.org/publications/RM/baran.list.html.
[8] B. A. Huberman, P. L. T. Pirolli, J. E. Pitkow, and R. J. Lukose, Science **280**, 95 (1998).
[9] J. Laherrere and D. Sornette, Eur. Phys. J. B **2**, 525 (1998).
[10] S. Redner, Eur. Phys. J. B **4**, 131 (1998).
[11] L. A. N. Amaral, A. Scala, M. Barthelemy, and H. E. Stanley, Proc. Natl. Acad. Sci. U.S.A. **97**, 11 149 (2000).
[12] M. E. J. Newman, Proc. Natl. Acad. Sci. USA **98**, 404 (2001).
[13] M. E. J. Newman, e-print cond-mat/0011144.
[14] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, Nature (London) **407**, 651 (2000).
[15] In the recent physical literatute, the term ''connectivity'' was frequently used for a number of connections of a node. This term has quite a different meaning in mathematical language. Here, stimulated by one of our previous referees, we use the mathematically correct term *degree*.
[16] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[17] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[18] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[19] A.-L. Barabási, R. Albert, and H. Jeong, Physica A **272**, 173 (1999).
[20] S. N. Dorogovtsev and J. F. F. Mendes, Phys. Rev. E **62**, 1842 (2000).
[21] H. A. Simon, Biometrika **42**, 425 (1955); *Models of Man* (Wiley, New York, 1957).
[22] S. Solomon and M. Levy, Int. J. Mod. Phys. C **7**, 745 (1996); D. Sornette and R. Cont, J. Phys. I **7**, 431 (1997); H. Takayasu, A.-H. Sato, and M. Takayasu, Phys. Rev. Lett. **79**, 966 (1997); M. Marsili, S. Maslov, and Y.-C. Zhang, Physica A **253**, 403 (1998).
[23] G. Bianconi and A.-L. Barabási, e-print cond-mat/0011224.
[24] S. M. Maurer and B. A. Huberman, e-print nlin.CD/0003041.
[25] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000); R. Albert and A.-L. Barabási, Phys. Rev. Lett. **85**, 5234 (2000).
[26] C. Moore and M. E. J. Newman, Phys. Rev. E **61**, 5678 (2000).
[27] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[28] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, e-print cond-mat/0010251.
[29] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, e-print cond-mat/0011115.
[30] P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[31] B. Bollobás, *Modern Graph Theory* (Springer, New York, 1998).
[32] J. Kleinberg, R. Kumar, P. Raphavan, S. Rajagopalan, and Tomkins (unpublished).
[33] S. Bornholdt and H. Ebel, e-print cond-mat/0008465.
[34] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, e-print cond-mat/0009090.
[35] S. N. Dorogovtsev and J. F. F. Mendes, Europhys. Lett. **52**, 33 (2000).
[36] S. N. Dorogovtsev and J. F. F. Mendes, Phys. Rev. E **63**, 025101 (2001); e-print cond-mat/0009065.
[37] G. Bianconi and A.-L. Barabási, e-print cond-mat/0011029.
[38] P. L. Krapivsky and S. Redner, e-print cond-mat/0011094.
[39] L. A. Adamic and B. A. Huberman, Science **287**, 2115a (2000).